Compound Poisson cascades

P. Chainais , R. Riedi , P. Abry

From self-similarity to infinitely divisible cascades

$$\delta_\tau X(t) = X(t+\tau) - X(t)$$

☛ SELF-SIMILARITY (stat. inc.) $\quad \mathbb{E}\,|\delta_\tau X|^q = \mathbb{E}\,|X(1)|^q \cdot |\tau|^{qH}$

ex : f.B.m., Linear Fractional Stable Motion...

☛ MULTIFRACTAL FORMALISM $\quad \mathbb{E}\,|\delta_\tau X|^q = c_q\,\tau^{\zeta(q)}$

ex : binomial cascades, wavelet random cascades, multifractal random walk (MRW)...

☛ LOG. INF. DIV. CASCADES (LIDC) $\quad \mathbb{E}\,|\delta_\tau X|^q = c_q \exp[-H(q)\cdot n(\tau)]$

- $H(q) \equiv \zeta(q)$ when $n(\tau) = -\log\tau$, (scale invariance)
- a priori $n(\tau) \neq -\log\tau$.

ex : ...

How to synthesize infinitely divisible cascades ?

Purpose : synthesis of processes with

- ☛ prescribed multiscaling exponents $H(q)$,
- ☛ continuous scale invariance, (not only 2^j)
- ☛ stationary increments,
- ☛ not a priori a power law : $\tau^{\zeta(q)} \longrightarrow \exp[-H(q) \cdot n(\tau)]$

Log-infinitely divisible cascades

$$\mathbb{E}\,|\delta_\tau X|^q \;=\; \exp[-H(q)\cdot n(\tau)]\;\cdot\; c_q$$

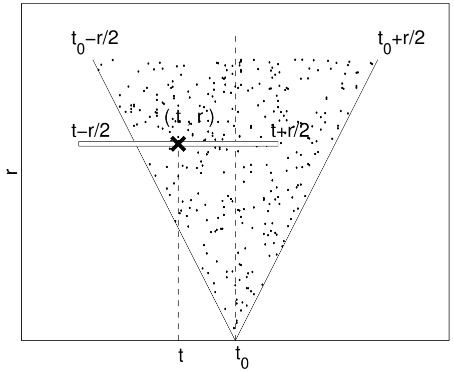
$$\Pi_{\log|\delta_\tau X|} \;=\; \underbrace{G^{*n(\tau)}}_{\text{propagator}} \;*\; \Pi_{\log|\delta_{\bar{T}} X|}$$

$$|\delta_\tau X| \;\overset{d}{=}\; \underbrace{[\Lambda^{(1)}\cdot\ldots\cdot\Lambda^{(n)}]}_{\text{(virtual) multipliers}} \;\cdot\; |\delta_{\bar{T}} X|$$

N.B. : $\Gamma_n = \prod_i \Lambda^{(i)}$ has a priori NO PHYSICAL MEANING (cf. fBm)

Compound Poisson cascades

$$\underbrace{G^{*[n(\tau_2)-n(\tau_1)]}(\log\Gamma)}_{G^{*\Delta n}} = e^{-\Delta n}\sum_{k=0}^{\infty}\frac{\Delta n^k}{k!}F^{*k}(\log\Gamma)$$



with $\log\Gamma = \log W_1 + \ldots + \log W_k$

- $\log W_i$ is distributed by F,
- k is Poisson distributed by $\mathcal{P}_{\Delta n}(k) = e^{-\Delta n}\dfrac{\Delta n^k}{k!}$

Poisson cascading noise

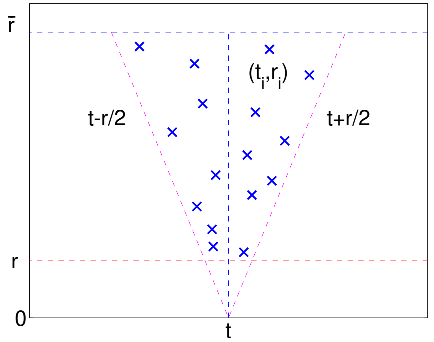
$$\Longrightarrow Q_r(t) \propto \prod_{\mathcal{C}_r(t)} W_i$$

MULTIPLICATIVE CASCADE

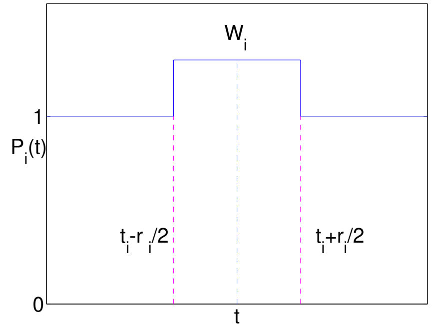
$(t_i, r_i)_{i \in I}$ = planar point process

$dm(t,r) = g(r)\, dr\, dt$ stationary,

W_i i.i.d., > 0, $F(\log W_i)$.

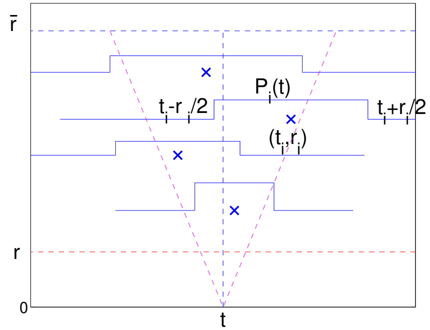


$$P_i(t) = \begin{cases} 1 & \text{if } |t - t_i| \geq \dfrac{r_i}{2}, \\ W_i & \text{if } |t - t_i| < \dfrac{r_i}{2}. \end{cases}$$



Poisson cascading noise

$$
\begin{aligned}
Q_r(t) &= \exp\left[(1-\mathbb{E}W)m(\mathcal{C}_r(t))\right] \prod_{r<r_i<1} P_i(t) \\
&= \exp\left[(1-\mathbb{E}W)m(\mathcal{C}_r(t))\right] \prod_{\mathcal{C}_r(t)} W_i
\end{aligned}
$$



$$\Downarrow$$

$$\mathbb{E}Q_r^q = \exp[-\varphi(q) \cdot m(\mathcal{C}_r)]$$

$$\varphi(q) = q(1-\mathbb{E}W) + \mathbb{E}W^q - 1,$$

where

$$\mathcal{C}_r(t) = \left\{(t', r') \,:\, r' > r \text{ and } t' - \frac{r'}{2} < t < t' + \frac{r'}{2}\right\}.$$

Poisson cascading noise

- $Q_r(t)$ is stationary,
- $\mathbb{E}\, Q_r(t) \;=\; 1,$
- $\mathbb{E} Q_r^q = \exp[-\varphi(q) \cdot m(\mathcal{C}_r)]$
- Q_r is a T-martingale (as $r \to 0$)

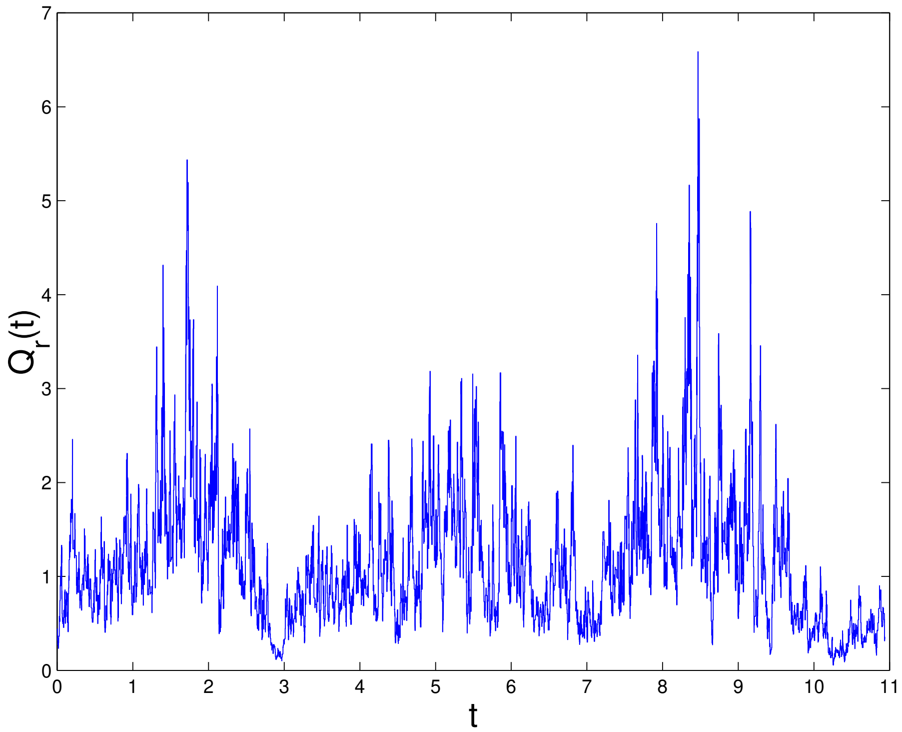
SCALE INVARIANCE $\quad n(\tau) = -\log(\tau)$

$$\Downarrow$$

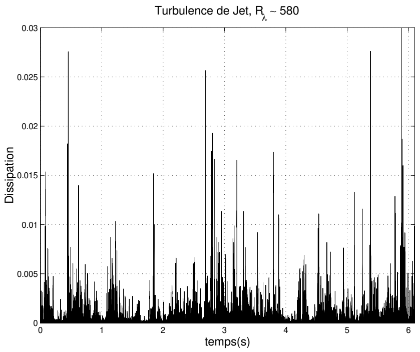
$$dm(t,r) = \frac{c}{r^2}\, dr\, dt \Longrightarrow \mathbb{E} Q_r^q = r^{c\varphi(q)}$$

Example of Poisson cascading noise

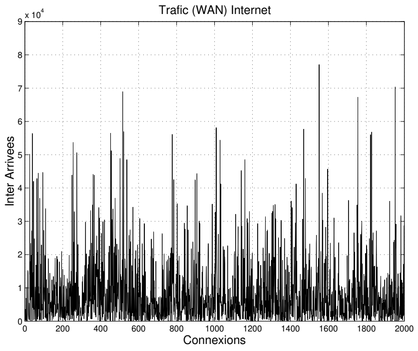
Poisson cascading noise



TURBULENCE



INTERNET



Poisson cascading motion

Pb : Q_r degenerates as $r \longrightarrow 0\ldots$

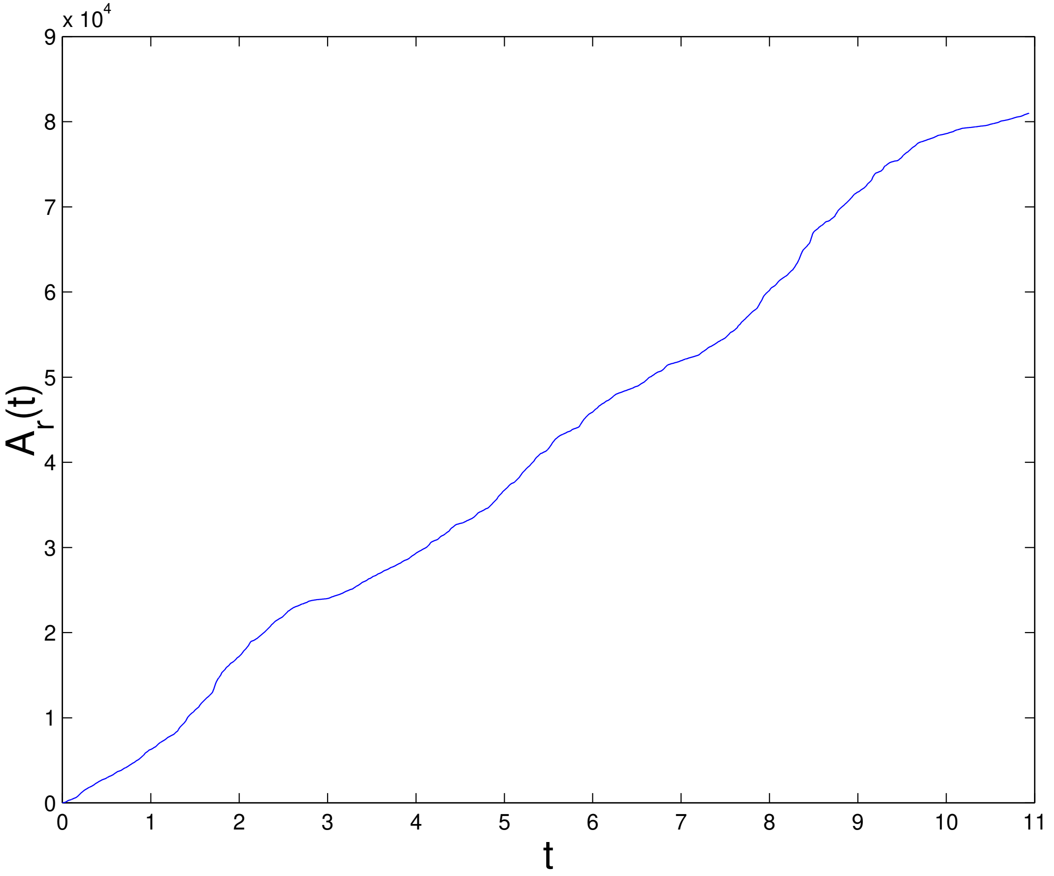
Solution : $$A_r(t) \;=\; \int_0^t Q_r(s)\,ds \quad\Longrightarrow\quad \mathbb{E}A_r(t) = t$$

$$A(t) \;=\; \lim_{r\longrightarrow 0} A_r(t)$$

has stationary increments

$$dm(t,r) = \frac{c}{r^2}\,dr\,dt \quad\Longrightarrow \mathbb{E}A(t)^q \sim t^{q+c\varphi(q)}$$

Poisson cascading motion



Poisson cascading Brownian motion

$$V_r(t) = \int_0^t \underbrace{\sqrt{Q_r(s)}\, dB(s)}_{\text{Poisson cascading Gaussian noise}}$$

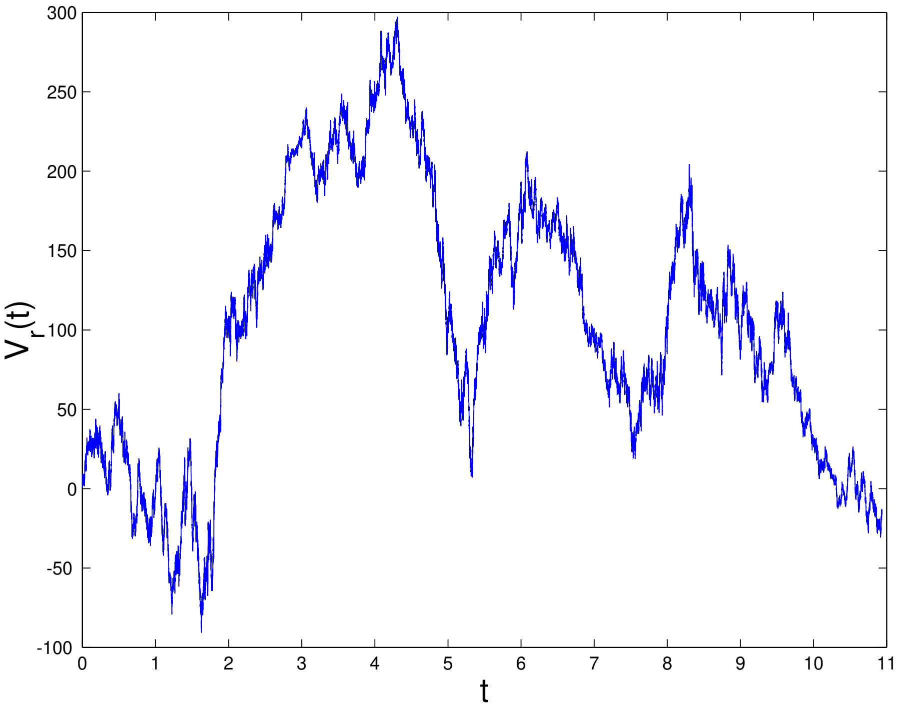
$$\Downarrow$$

$$V(t) = \lim_{r \to 0} V_r(t)$$

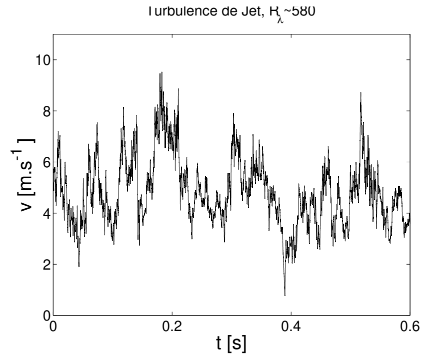
- stationary increments
- $dm(t,r) = \frac{c}{r^2}\, dr\, dt \Longrightarrow \mathbb{E}V(t)^q \sim t^{q/2 + c\varphi(q/2)}$
  continuous scaling
- $V_r(t) \overset{fdd}{=} B(A_r(t))$ (cf. fBm in multifractal time)

Example of Poisson cascading Brownian motion

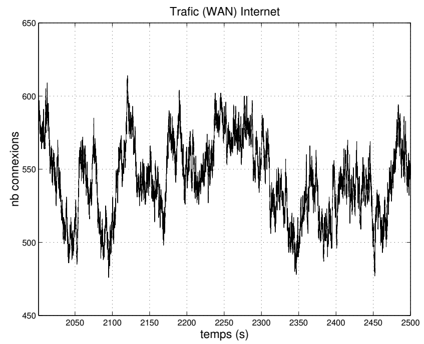
Poisson cascading Brownian motion



TURBULENCE



INTERNET



Conclusion

POISSON CASCADING... ...HAS... ...NICE PROPERTIES

• noise	$\mathbb{E}Q_r(t) = r^{c\varphi(q)}$	• stationary increments,
• motion	$\mathbb{E}A(t) \sim t^{q+c\varphi(q)}$	• continuous scale invariance,
• Brownian motion	$\mathbb{E}V(t) \sim t^{q/2+c\varphi(q/2)}$	• continuous in time ($r \to 0$).

• $\mathbb{E}Q_r(t) = \exp[-\varphi(q) \cdot m(\mathcal{C}_r)]$

log-infinitely divisible $\quad dm(t,r) \neq \frac{c}{r^2} dr dt$

(not compound Poisson a priori) $\quad$ towards $n(\tau) \neq -\log \tau \ldots$

• $V_r(t) = \int_0^t \sqrt{Q_r(s)}\, dB_H(s) \Rightarrow$ cascading *fractional* Brownian motion...

fractional stable motion...